Network Representation for Lossless Symmetrical Discontinuities in a Multimode Waveguide

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Abstract—It is shown that a symmetrical, lossless $2N$-port may be represented by an equivalent circuit, a $2N$-port lattice network, which is a $2N$-port generalization of the known two-port symmetrical lattice. The approach employed here in finding the equivalent circuits is based on the fact that any symmetrical $2N$-port may be decomposed into two general N-port networks, each of which represents the opencircuit or short-circuit bisected structures.

I. Introduction

An EQUIVALENT circuit has been found for a symmetric, lossless $2N$-port network which is often encountered in connection with discontinuity structures in a multimode waveguide. The physical configuration may involve either conducting obstacles in the waveguide, transverse apertures, or apertures in the waveguide wall, and the structure is assumed to be symmetric in the sense that there exists a transverse plane of bisection. While equivalent circuits for arbitrary two or three ports can easily be constructed in terms of either the admittance or the impedance matrix elements, the equivalent circuits for networks with more than three ports, to be introduced in the following, are not generally available in the literature.

For a symmetrical structure, the overall scattering, impedance or admittance matrix is found to be comprised of the sum of two matrices fixing the responses to symmetrical and antisymmetrical excitation, respectively. The constraint of conservation of energy is therefore applicable separately to the parameters representative of symmetric and of antisymmetric excitation, or alternatively to the open-circuit short-circuit bisected structures [1]. The general approach employed here in finding the equivalent circuits is based on the fact that any symmetrical $2N$-port may be decomposed into two general N-port networks, each of which represents the open-circuit or short-circuit bisected structures. The resultant equivalent network is a $2N$-port generalization of the known two-port symmetrical lattice structure, and will be called the $2N$-port lattice network. The equivalent circuits of the general lossless N-ports occurring in the bisected structures are presented in terms of the admittance matrix elements [2].

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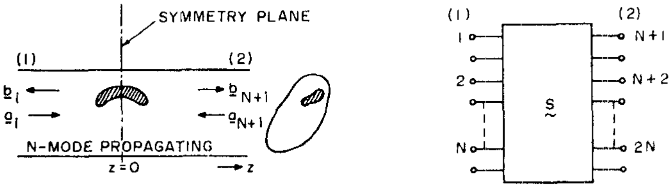
II. Network Analysis for Lossless, Symmetric Multimode Discontinuities

A. The Basic Problem

Consider the configuration in Fig. 1 where a perfectly conducting symmetric obstacle is situated in a uniform waveguide bounded by perfectly conducting walls and is capable of propagating N modes. The characteristics of the far fields of any structure (involving propagating modes only) are describable in terms of either standing-wave or traveling-wave amplitudes at modal reference planes chosen at a distance several wavelengths away from the obstacle plane $z=0$. Such formulations give rise to impedance matrix or scattering matrix descriptions of the $2N$-port network equivalent of the structure.



(a) (b)

Fig. 1. Symmetric multimode discontinuity. (a) Physical structure. (b) Equivalent network.

B. Scattering Matrix for Symmetrical 2N-Port Structures

Let S be the scattering matrix defined as

$$b = \mathsf{S}a, \qquad a = \begin{pmatrix} a^{(1)} \\ a^{(2)} \end{pmatrix}, \qquad b = \begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix}, \qquad (1)$$

where $a^{(1)}$ and $a^{(2)}$ are the incident mode vectors for the electric field (voltages) at terminals (1) and (2), respectively. $b^{(1)}$ and $b^{(2)}$ are the corresponding reflected mode vectors (Fig. 1). For a general $2N$-port structure with reflection symmetry as portrayed in Fig. 1, S must satisfy $\mathsf{S}=\tau\mathsf{S}\tau^{-1}$ [3] where

$$\tau = \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix}_{2N\times 2N}$$

and 1_N is the $N\times N$ unit matrix. Thus, S may be written as

$$\mathsf{S} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{B} & \mathsf{A} \end{pmatrix}_{2N\times 2N}, \qquad (2)$$

with $\mathbf{A}$ and $\mathbf{B}$ representing $N\times N$ submatrices. $\mathbf{S}$ in (2) may now be decomposed into two partial matrices, one corresponding to symmetrical, and the other to antisymmetrical, excitation. The same decomposition may be achieved by the bisection theorem [1].

Let us write $\boldsymbol{a}$ of (1) as a sum of symmetrical and antisymmetrical excitations, which are denoted by superscript e or o, respectively;

$$\boldsymbol{a} = \boldsymbol{a}^e + \boldsymbol{a}^o \tag{3}$$

where

$$\boldsymbol{a}^{\overset{e}{o}} = \boldsymbol{\tau}^{\overset{e}{o}}\boldsymbol{a}, \qquad \boldsymbol{\tau}^{\overset{e}{o}} = \frac{1}{2}\begin{pmatrix} 1_N & \pm 1_N \\ \pm 1_N & 1_N \end{pmatrix}_{2N\times 2N} \tag{3a}$$

By substituting (3) into (1) and operating in (1) with $\boldsymbol{\tau}^e$ and $\boldsymbol{\tau}^o$ defined in (3a), we get

$$\mathbf{S} = \mathbf{S}_e + \mathbf{S}_o, \qquad \mathbf{S}_{\overset{e}{o}} = \frac{1}{2}\begin{pmatrix} \mathbf{s}^{\overset{e}{o}} & \pm \mathbf{s}^{\overset{e}{o}} \\ \pm \mathbf{s}^{\overset{e}{o}} & \mathbf{s}^{\overset{e}{o}} \end{pmatrix}_{2N\times 2N} \tag{4}$$

with $\mathbf{s}^e$ and $\mathbf{s}^o$ denoting $N\times N$ submatrices and defined by $\mathbf{s}^e = \mathbf{A}+\mathbf{B}$ and $\mathbf{s}^o = \mathbf{A}-\mathbf{B}$. $\mathbf{S}_e$ and $\mathbf{S}_o$ give rise to the even and odd responses, i.e., $\boldsymbol{b}^{\overset{e}{o}} = \mathbf{S}_{\overset{e}{o}}\boldsymbol{a}$.

C. *Decoupling of the Partial Networks*

If we apply the condition of energy conservation in a lossless network to $\mathbf{S}$ of (4), $\mathbf{S}\mathbf{S}^* = 1_{2N}$, we may verify that

$$\mathbf{s}^e\mathbf{s}^{e*} = 1_N$$
$$\mathbf{s}^o\mathbf{s}^{o*} = 1_N. \tag{5}$$

The conditions obtained in (5) highlight the decoupling of the overall structure into even and odd partial networks since $\mathbf{s}^e$ and $\mathbf{s}^o$ are actually the $N\times N$ scattering matrices appropriate to a general lossless N-port network.

III. Equivalent Circuit Representation for Symmetrical 2N-Port Structures

A. *The Equivalent Circuit*

For a representation in terms of an equivalent network containing conventional circuit elements, it is useful to obtain the impedance or admittance matrix via

$$\mathbf{Z} = (1_{2N} + \mathbf{S})(1_{2N} - \mathbf{S})_{-1} \text{ or } \mathbf{Y} = (1_{2N} - \mathbf{S})(1_{2N} + \mathbf{S})^{-1}.$$

For networks defined by (2) or (4), we get

$$\mathbf{Z} = \mathbf{Z}_e + \mathbf{Z}_o \qquad \mathbf{Y} = \mathbf{Y}_e + \mathbf{Y}_o \tag{6}$$

$$\mathbf{Z}_{\overset{e}{o}} = \frac{1}{2}\begin{pmatrix} \mathbf{z}^{\overset{e}{o}} & \pm \mathbf{z}^{\overset{e}{o}} \\ \pm \mathbf{z}^{\overset{e}{o}} & \mathbf{z}^{\overset{e}{o}} \end{pmatrix} \quad \mathbf{Y}_{\overset{e}{o}} = \frac{1}{2}\begin{pmatrix} \mathbf{y}^{\overset{e}{o}} & \pm \mathbf{y}^{\overset{e}{o}} \\ \pm \mathbf{y}^{\overset{e}{o}} & \mathbf{y}^{\overset{e}{o}} \end{pmatrix}. \tag{7}$$

Here

$$\mathbf{z}^{\overset{e}{o}} = \mathbf{y}^{\overset{e}{o}\,-1} = (1_N + \mathbf{s}^{\overset{e}{o}})(1_N - \mathbf{s}^{\overset{e}{o}})^{-1}. \tag{8}$$

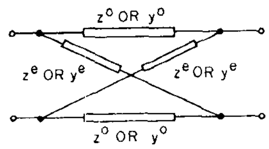


Fig. 2. Lattice network.

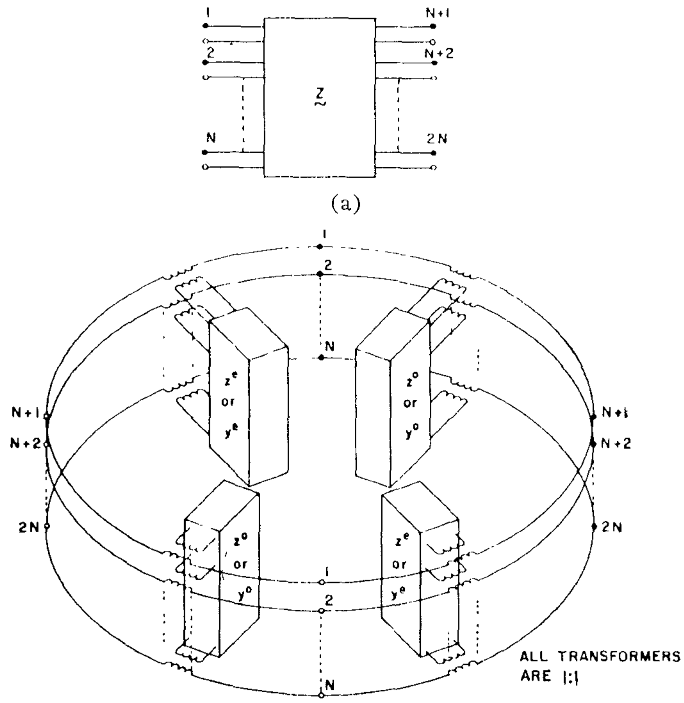


Fig. 3. Equivalent circuit for symmetric $2N$-port network.

Evidently the partial matrices $\mathbf{z}^e$ and $\mathbf{z}^o$, or $\mathbf{y}^e$ and $\mathbf{y}^o$, may be realized separately as two general lossless N-port networks which again satisfy the condition of conservation of energy.

To construct an equivalent circuit involving the open- and short-circuit partial matrices in (8), we recall the impedance and admittance matrices of a two-port lattice network, which is shown in Fig. 2:

$$\mathbf{Z} = \frac{1}{2}\begin{pmatrix} z^e+z^o & z^e-z^o \\ z^e-z^o & z^e+z^o \end{pmatrix}, \quad \mathbf{Y} = \frac{1}{2}\begin{pmatrix} y^e+y^o & y^e-y^o \\ y^e-y^o & y^e+y^o \end{pmatrix}. \tag{9}$$

One observes from (6), (7), and (9) that the coupling between any two ports of a $2N$-port symmetrical structure, when the remaining ports are all open-circuited or short-circuited, respectively, is similar to that of the two-port lattice network portrayed in Fig. 2. It has been found that, by utilizing the basic lattice coupling characteristics, the equivalent circuit for a symmetrical $2N$-port can be constructed as shown in Fig. 3. Such an equivalent circuit, which will be called $2N$-port lattice network of which the two-port lattice network is but a special case, is shown in the Appendix to represent a general $2N$-port symmetrical structure which is characterized as in (2) or (4).

As is well known, a general lossless reciprocal N-mode junction or $2N$-port network must have in its network description a number of independent real parameters (scattering coefficient amplitudes and phases, impedance matrix elements, etc.) given by $V=N(2N+1)$, where N is the number of propagating modes coupled by the discontinuity. If the junction is also symmetrical, V is reduced to $V=N(N+1)$ which is exactly the sum of the number of independent elements in the reciprocal N-port partial networks, $\mathbf{z}_o^e$ or $\mathbf{y}_o^e$ appearing in the $2N$-port lattice structure.

B. *Equivalent Circuits for 2N-Port Pure Shunt and Pure Series Networks*

Although we have shown the equivalent circuit in its most general form, it is of interest to point out two special cases which occur frequently in various problems.

First we consider a symmetrical $2N$-port structure which reduces to a short circuit for antisymmetrical excitation. This implies that $\mathbf{S}_o=-\boldsymbol{\tau}^o$ [(3a) and (4)], and one may derive directly from (6)–(8) that the admittance matrix for this case does not exist ($\mathbf{Y}_o\rightarrow\infty$) while the impedance matrix reduces to $\mathbf{Z}=\mathbf{Z}_e(\mathbf{Z}_o=0)$. The equivalent circuit for such a structure is apparently a pure shunt network which may be obtained directly from Fig. 3 by short-circuiting all of the N ports of the partial networks $\mathbf{z}^o$. The resultant equivalent circuit is identical to that proposed by Felsen, Kahn and Levey [4] (Fig. 4).

Alternatively, if the given structure becomes open-circuited when excited symmetrically, we have instead the relation $S_e=\tau^e$ which leads to an infinite impedance matrix ($\mathbf{Z}_e\rightarrow\infty$). In this case, the network property can only be defined by an admittance matrix $\mathbf{Y}=\mathbf{Y}_o(\mathbf{Y}_e=0)$ which implies that all of the N ports of the partial networks $\mathbf{y}^e$ in Fig. 3 are open-circuited. We obtain thereby the equivalent circuit for $2N$-port pure series structures (Fig. 5).

IV. Application to a Two-Mode Symmetrical Discontinuity Structure

As an illustration, let us consider the case of a symmetrical, lossless discontinuity in a waveguide propagating two modes. We assume that both modes are coupled through the discontinuity structure. The equivalent circuit representation for a two-mode coupling structure may be obtained directly from Fig. 3 and is shown as a four-port lattice network in Fig. 6 where $\mathbf{z}^e$ and $\mathbf{z}^o$ (or $\mathbf{y}^e$ and $\mathbf{y}^o$) are represented conveniently by Weissfloch two ports [5]. This equivalent circuit is useful for the measurement of two-mode symmetrical discontinuities by a resonant cavity technique [6] if the cavity may be excited in the open-circuit or short-circuit bisection modes, or alternatively, if the resonance corresponds to symmetrical or antisymmetrical excitation (this may be achieved, for example, by symmetrical motion of two movable end plates of the cavity) [6].

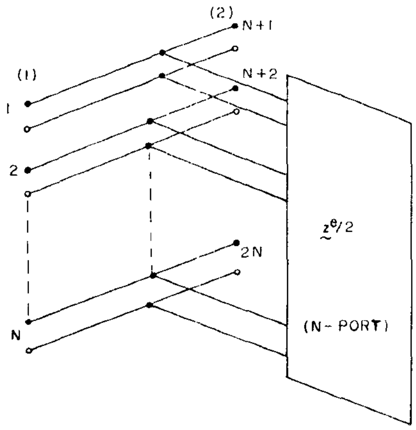


Fig. 4. Equivalent circuit for $2N$-port pure shunt structures.

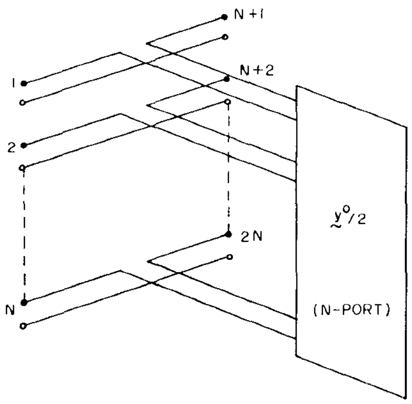


Fig. 5. Equivalent circuit for $2N$-port pure series structures.

For symmetrical excitation [Fig. 7(a)], the equivalent circuit in Fig. 6(b) is reduced to Fig. 7(b) and the problem is thus simplified to measuring the two port $\mathbf{z}^e$ or $\mathbf{y}^e$ as portrayed in Fig. 7(c); the Weissfloch transformed network is convenient for a systematic analysis of multipoint data taken on such a structure.

Similarly, for antisymmetrical excitation [Fig. 8(a)], the measurement is reducible to that in Fig. 8(c).

Appendix

Generality of the 2N-Port Lattice Network

To show the generality of the $2N$-port lattice network, we must first verify that an open- or short-circuit occurs at the network ports of the partial networks $\mathbf{z}_o^e$ or $\mathbf{y}_o^e$ when we open-circuit or short-circuit the corresponding ports of the $2N$-port lattice network. Let us connect a load impedance Z_L to all but the teriminals of mode i of the $2N$-port lattice network, so that the resultant circuit is as shown in Fig. 9(a). From simple symmetry arguments, one finds that the voltage and current distribution at the load of each mode will be as indicated in Fig. 9(b), from which it is easy to see that the load impedance of the partial networks $\mathbf{z}_o^e$ or $\mathbf{y}_o^e$ for each of the N modes is Z_L; i.e., $V_j^e/i_j^e=V_j^o/i_j^o=Z_L$.

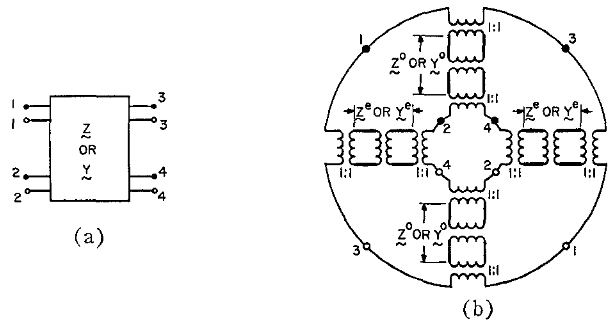


Fig. 6. Four-port lattice network.

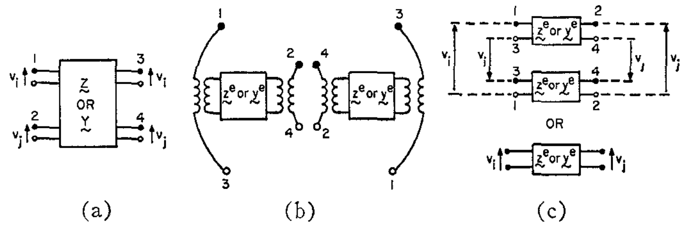


Fig. 7. Partial network for symmetrical excitation.

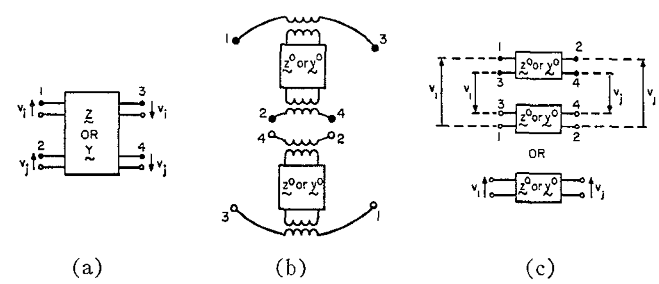


Fig. 8. Partial network for antisymmetrical excitation.

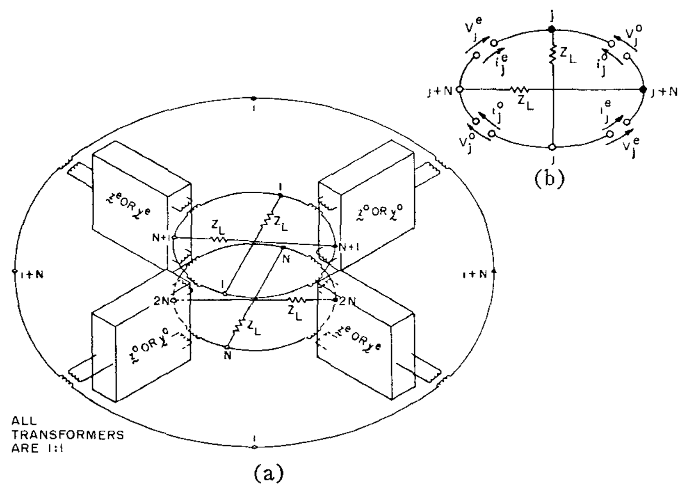


Fig. 9. Definition of the partial networks $\mathbf{z}^o$ and $\mathbf{y}^o$.

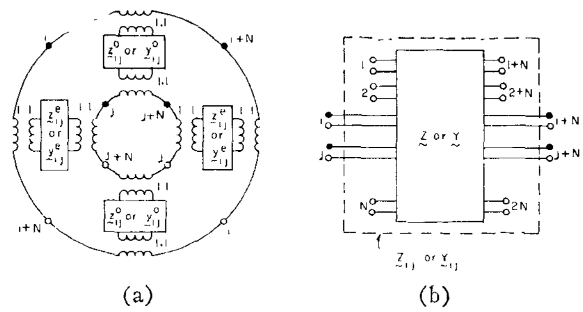


Fig. 10. Equivalent circuit for the submatrices of Z or Y with indices i and j.

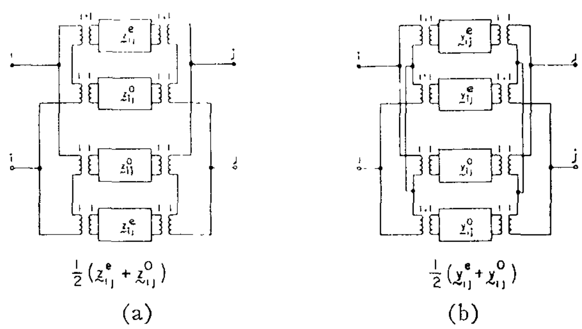


Fig. 11. Equivalent circuits for diagonal submatrices. (a) $(\mathbf{z}_{ij}^e+\mathbf{z}_{ij}^o)/2$. (b) $(\mathbf{y}_{ij}^e+\mathbf{y}_{ij}^o)/2$.

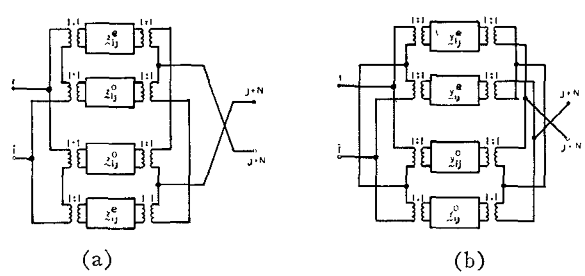


Fig. 12. Equivalent circuits for off-diagonal elements of $(\mathbf{z}_{ij}^e-\mathbf{z}_{ij}^o)/2$ and $(\mathbf{y}_{ij}^e-\mathbf{y}_{ij}^o)/2$.

Therefore, $\mathbf{z}_o^e$ or $\mathbf{y}_o^e$ in Fig. 3 is indeed the open-circuit impedance ($Z_L=\infty$) or short-circuit admittance matrix ($Z_L=0$), respectively.

Next, we will show the validity of the equivalent circuit of Fig. 3 as regards the lattice coupling prescribed in (6) and (7). To do this, let us assume that all but the network ports of the ith and jth modes are open- or short-circuited. The $2N$-port lattice network thus reduces to the four-port lattice network shown in Fig. 10. The diagonal submatrices of $\boldsymbol{Z}$ or $\boldsymbol{Y}$ with mode indices i and j, $(\boldsymbol{z}_{ij}^e+\boldsymbol{z}_{ij}^o)/2$ or $(\boldsymbol{y}_{ij}^e+\boldsymbol{y}_{ij}^o)/2$ as defined is (6) and (7), respectively, are, by definition, the impedance matrix or admittance matrix for the network ports i and j when $i+N$ and $j+N$ are open-circuited or short-circuited, respectively [Fig. 10(b)]. If the network shown in Fig. 10(a) is indeed the equivalent of that in Fig. 10(b), we should obtain an identical representation of the impedance or admittance matrix between their corresponding ports. This may be verified from the network of Fig. 11, which is obtained from the four-port lattice network of Fig. 10(a) with its network ports $i+N$ and $j+N$ open-circuited or short-circuited, respectively. It is found that the impedance matrix of Fig. 11(a) is $(\boldsymbol{z}_{ij}^e+\boldsymbol{z}_{ij}^o)/2$ while the admittance matrix of Fig. 11(b) is $(y_{ij}^e+y_{ij}^o)/2$. The connection of one-to-one ideal transformers in the equivalent circuit of Fig. 10 (and Fig. 3) is necessary since the flow of loop currents between the partial networks z_{ij}^{oe} or y_{ij}^{oe} in Fig. 11 must be prevented.

To conclude our proof, the off-diagonal submatrices $(z_{ij}^e-z_{ij}^o)/2$ or $(y_{ij}^e-y_{ij}^o)/2$ still remain to be identified. The diagonal elements of $(z_{ij}^e-z_{ij}^o)/2$ are, by definition, the transfer impedance between port i and $i+N$, or j and $j+N$, in Fig. 10(b). Here we notice that the difference between the corresponding diagonal elements of $(z_{ij}^e-z_{ij}^o)/2$ and of $(z_{ij}^e+z_{ij}^o)/2$ is merely the sign between the partial networks z_{ij}^e and z_{ij}^o. This is shown clearly in Fig. 10 where i and $i+N$ are the network ports of a two-port lattice network when j and $j+N$ are open-circuited and vice versa. Similarly, Fig. (12a) may be employed to show that the off-diagonal elements of $(z_{ij}^e-z_{ij}^o)/2$ in the equivalent circuit of Fig. 10(a)are the transfer impedances between the corresponding network ports of Fig. 10(b). Analogous considerations apply to the identification of $(y_{ij}{}^e_o-y_{ij}{}^e_o)/2$ as the transfer admittances between the network ports of Fig. 10(a) [see Fig. 11(b)].

Since the mode indices i and j are chosen arbitrarily in the preceding discussion, the proof of generality for the $2N$-port lattice network is complete. Thus, we may conclude that the network given in Fig. 3 is capable of representing any lossless, symmetrical, $2N$-port structure characterized as in (2) or (4).

Acknowledgment

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Perturbation Theorems for Waveguide Junctions, with Applications

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Abstract—**Perturbation theorems are derived in the context of a theory of waveguide junctions. These theorems express changes in impedance or admittance matrix elements, due to changes in a waveguide junction, in terms of integrals over products of perturbed and unperturbed basis fields associated with the junction and with its adjoint. Media involved are required only to be linear.**

Concepts of first-order perturbation theory are discussed briefly, and the term "correct to the lowest order" is precisely defined. The need of explicit theorems telling when one may expect results actually correct to the lowest order is noted.

Two problems are solved approximately by the perturbation approach:

1) reflection at the junction of rectangular waveguide with filleted waveguide of the same main dimensions; and

2) the effect of finite conductivity of both obstacle and waveguide wall for half-round inductive obstacles in rectangular waveguide.

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I. Introduction

THE PURPOSE of this paper is to present certain perturbation theorems in the context of a theory of waveguide junctions, to discuss briefly some of the rationale and the peculiarities of the simplest applications of perturbation methods, and to solve several problems that are illustrative as well as useful.

The presentation of the theorems in Section III of this paper was inspired largely by a paper by Monteath,[1] which gives theorems of the same type, but in a

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